ANALYSIS OF A CYLINDRICAL ACOUSTIC RESONATOR SOUND-VELOCITY METER

James E. Wille

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ACO - ACOUSTIC
VEL - VELOCITY
MEU - METER

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Lieutenant, United States Navy

Submitted in partial fulfillment of
the requirements for the degree of

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IN

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ABSTRACT

A cylindrical acoustic resonator which has been developed by Dr. William J. Toulis of the Signal Propagation Division of the United States Navy Electronic Laboratory for measurement of the velocity and absorption characteristics of an enclosed fluid medium is discussed. The cavity is constructed of compliant tubes, thus presenting a pressure release surface at the boundaries. Theory is developed indicating that the resonant frequency of the cavity is directly proportional to the velocity of sound in the medium while the Q of the resonator can be used to determine attenuation.

The writer wishes to express his appreciation for the guidance and encouragement of Professor Herman Medwin of the United States Naval Postgraduate School. He further wishes to acknowledge the assistance given him by Dr. Toulis and other members of the Signal Propagation Division of the United States Navy Electronics Laboratory in this investigation.

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TABLE OF SYMBOLS

a - cylinder radius

$\bar{A}$ - vector potential

B - bulk modulus of elasticity

c, c_o - speed of sound, reference speed

C_p, C_v - specific heats at constant pressure, constant volume

d_{ij} - rate of deformation tensor

D - material differential operator

E_T, E_I, E_K - total energy, internal energy, kinetic energy

f - frequency

$\bar{F}$ - extraneous force

G - $\frac{N \nu \omega}{\rho_o c_o^2}$, a frequency dependent ratio

H - $\frac{\kappa}{\nu N C_p}$, non-frequency dependent ratio

i - designation of imaginary axis

K - phase constant - $\frac{\omega}{c}$

L - cylinder length

ℓ - number of pressure antinodes in the axial direction

N - viscosity number = $N = 2 + \lambda/\nu$

P; P_1, P_2 - incremental, or sound, pressures; first and second order pressures

$\bar{q}$ - heat flux vector

$\mathcal{Q}$ - Stokes radiation coefficient (see pp. 8)

Q - quality factor

$\bar{S}$ - closed surface

s - specific entropy

T - absolute temperature

t - time

t_{ij} - stress tensor

u_{ij} - residual stress

V - volume

$\bar{V}$ - particle velocity

υ - specific volume

W - energy stored

α - attenuation constant

β - coefficient of thermal expansion

γ - ratio of specific heats - C_p/C_v

δ_{ij} - Kronecker delta

ε - specific internal energy per unit mass

θ_1 - first-order variational component of temperature

K - thermal conductivity

λ - dilatational viscosity coefficient

μ - dynamic shear viscosity coefficient

ρ; $\rho_1 \rho_2$ - total density; first and second order variational components

σ - time dependence variable ($\sigma = \lambda\alpha + \omega$)

ϕ - scalar velocity potential

Φ_n - viscous dissipation function

ω - angular frequency

S_{mn} - nth zero of Bessel function of order m

∇ () - gradient

$\nabla \cdot$() - divergence

∇ X() - curl

1. Introduction.

The success of sonic echo-ranging methods depends, in part, upon the velocity and attenuation profiles in the transmitting medium between the detecting ship and the target. The development of an instrument which could directly measure these properties of the sea would be of great assistance.

The velocity of sound in sea water under varying physical conditions has been investigated both experimentally and theoretically by many persons. For the experimental determinations, much of the early work was pointed toward accurate determination of depth by sonic sounding methods. Transmission time between a transmitting and receiving hydrophone was measured in an area where salinity and temperature were as uniform as possible, and in deep enough water so that bottom reflections were minimized. In this type of determination, measurement of the distance is the limiting factor.

More recently, a number of velocity meters has been developed for this purpose. The most successful for laboratory use has been the interferometer, where a standing wave is set up in a cell by utilizing a crystal and reflector or two reflectors. This then provides a measure of the wave length and hence, the velocity. However, no completely satisfactory operational-type system is presently available.[1] The most widely used device to determine sound velocity profiles is the bathythermograph, which measures temperatures rather than velocities. Its use depends on the fact that the velocity of sound is largely determined by the temperature of the sea, and only to a lesser extent by salinity and pressure. Urick[2] obtained good general agreement between velocity-depth plots obtained with an interferometer and those computed from bathythermograph and thermocouple observations for sea states of zero to three and depths

[1]H. Sussman, Study and Evaluation of ONR-Sponsored Sound-Velocity Meters, USL Research Report No. 299, 1 March 1956

[2]R. J. Urick, An Acoustic Interferometer For the Measurement of Sound Velocity In the Ocean, USNR & S Lab. Report No. S-18

as great as 150 feet. However, the bathythermograph is insensitive to temperature microstructures,[3] and as such, does not give an accurate indication of transmission characteristics.[4]

A great deal of continuing effort has been given to the theoretical determination of the velocity of sound from the physical constants of the medium as given by the theory of thermodynamics.[5,6,7] Much of the work is, of necessity, empirical in nature and has been complicated by a lack of sufficiently accurate experimental data.[8] However, recent improvements in laboratory measuring techniques should rapidly improve this situation.[9]

Techniques for measuring attenuation of sound by sea water are even less satisfactory. The absorption at low frequencies is sufficiently small to make measurement difficult without considering a large expanse of sea. Here again, determination of distance is a limiting factor. Further isolation of the specific causes of attenuation is next to impossible without a device which can determine the absorption characteristics of small samples.

The velocity meter under discussion depends, for its operation, on the fact that the resonant frequency of a water column in a particular

[3]Urick, ibid.

[4]L. Liebermann, The Effect of Temperature Inhomogeneities in the Ocean On the Propagation of Sound, J. Ac. Soc. Am., 23, pp 563 (1951)

[5]S. Kuwahara, The Velocity of Sound in Sea Water and Calculation of the Velocity for Use in Sonic Sounding, Japanese J. Astron. & Geophys., Vol. 16, No. 1 (1938)

[6]D. J. Matthews, Tables of the Velocity of Sound in Pure Water and Sea Water for Use in Echo-Sounding & Sound-Ranging, Hydrographic Dept. Admiralty, London, H.D. 282 (1927)

[7]V. A. Del Grosso, The Velocity of Sound in Sea Water at Zero Depth, NRL Report 4002, 11 June 1952

[8]R. T. Beyer, Formulas for Sound Velocity in Sea Water, J. Mar. Research, Vol. 13, pp 113 (1954)

[9]V. A. Del Grosso, E. J. Smura and P. T. Fougere, Accuracy of Ultrasonic Interferometer Velocity Determinations, NRL Report 4439, 6 December 1954

mode of vibration is directly proportional to the velocity of sound in the water column. The technique employed is not new in itself. A hard wall cavity resonator was developed by the University of Michigan in 1952.[10] However, this device was extremely sensitive to wall cleanliness due to the relatively high sonic pressures at the boundaries, characteristic of rigid walled devices. Thus, although the instrument provides an accurate measure of the velocity when used in the laboratory, it is very difficult to use under operational conditions.

Use of a pressure release walled cylinder suggests itself; and, in fact, thin walled cylinders have been used for measurement of the attenuation of marine sediments.[11,12] Recent developments in the field of compliant structures by Dr. William J. Toulis of the Signal Propagation Division of the United States Navy Electronics Laboratory, San Diego, California, enabled him to adapt this technique to the present operational device, where, by measuring the resonant frequency and the "Q" of the cavity, the desired characteristics of a small sample of water can be determined to a high degree of accuracy.

The purpose of this paper is to discuss the theoretical aspects of the pressure release walled cylindrical cavity constructed by Toulis, including a somewhat systematic study of the loss problem.

[10] R. K. Brown, The Development of an Underwater Sound Velocity Meter Using a Cylindrical Resonator, Proj. 2021, final report, 30 April 1953, Eng. Research Inst. Univ. of Michigan

[11] W. J. Toulis, Theory of a Resonance Method to Measure the Acoustic Properties of Sediments, NEL Report 676, 5 March 1956

[12] G. Shumway, A Resonant Chamber Method for Sound Velocity and Attenuation Measurement in Sediments, Geophys. Vol XXI, No. 2, pp 305-319, April 1956

2. General Wave Equation.

Before proceeding to the actual analysis of the velocity meter, it is necessary to develop the equations governing the transmission of acoustic waves through a fluid medium. It is assumed that the medium is locally homogeneous, continuous except at the boundaries, and isotropic when at rest. It is further assumed that the condition of the medium can be fully described by establishing the functional dependence of pressure, vector velocity, density and temperature on time and the space coordinates. A set of equations consisting of

a) the continuity equation expressing the conservation of mass

b) a force equation based on conservation of momentum

c) a heat exchange equation based on conservation of energy

will be developed in their complete form to present a rigorous point of departure for making the necessary approximations for the study of small-signal effects.

To derive the first of the required relationships, consider an elemental volume V bounded by a closed surface $\overline{S}$. The amount of fluid crossing the surface in time Δt is given by the surface integral:

$$\text{outflow} = \Delta t \int_S \rho \overline{V} \cdot d\overline{S}$$

where

$\overline{V}$ = particle velocity

ρ = point fluid density

During this same period, the quantity of fluid originally contained in V will have diminished by the amount:

$$\text{loss} = -\Delta t \int_V \frac{\partial \rho}{\partial t} \, dV$$

where the negative sign is selected because density is a decreasing function of time. Since mass must be conserved, net outflow must equal the loss of mass and the two expressions can be equated. The surface integral is then converted by use of the divergence theorem giving:

$$\int_V \left[\nabla \cdot (\rho \overline{V}) + \frac{\partial \rho}{\partial t} \right] = 0$$

Since the integrand is continuous and the volume arbitrary, this may be written as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \overline{V} \right) = 0 \qquad 2.1$$

which is one form of the well-known equation of continuity.

Considering the density is a function of both space and time, the total derivitive of ρ with respect to t can be written in vector notation as:

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \bar{V} \cdot \nabla \rho \qquad 2.2$$

utilizing the notation of Stokes[13] where $\frac{D}{Dt}$ denotes the material derivitive and is defined by

$$\frac{D}{Dt} = \frac{\partial (\ \ )}{\partial t} + \bar{V} \cdot \operatorname{grad}(\ \ \ ).$$

Substituting from equation 2.1 into 2.2 gives

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \bar{V} = 0 \qquad 2.3$$

which is Euler's form of the continuity equation.

To develop the second of the basic relationships, consider Cauchy's first law of motion:[14]

$$\rho\left(\frac{Dv_\lambda}{Dt}\right) = \rho F_\lambda + \frac{\partial t_{\lambda\gamma}}{\partial x_\gamma} \qquad 2.4$$

where:

F_λ = extraneous force vector, a function of position and time

v_λ = velocity component of

$t_{\lambda\gamma}$ = stress tensor.

For an isotropic medium where stress is a linear function of the rate of deformation, the stress tensor can be resolved into a sum of a scalar and a residual stress tensor as follows:

$$t_{\lambda\gamma} = -p\delta_{\lambda\gamma} + u_{\lambda\gamma} \qquad (t_{\lambda\gamma} = t_{\gamma\lambda}) \qquad 2.5$$

where

p = pressure

$u_{\lambda\gamma}$ = residual stress

$\delta_{\lambda\gamma}$ = Kronecker delta

$p\delta_{\lambda\gamma}$ = hydrostatic tension.

[13] C. Truesdell, The Mechanical Foundations of Elasticity and Fluid Dynamics, J. Rat. Mech. & Anal., 1, pp 125, (1952)

[14] F. V. Hunt, Notes on the Exact Equations Governing The Propagation of Sound in Fluids, J. Acous. Soc. of Am., 27, pp 1022, Nov. 1955.

To a first approximation, the residual stress can be expressed by the linear terms of a power expansion in the viscosity coefficients as:

$$u_{ij} = \lambda d_{kk} \delta_{ij} + 2\mu d_{ij} \qquad 2.6$$

where:

d_{ij} = rate of deformation tensor

μ = dynamic shear viscosity coefficient

λ = dilatational viscosity coefficient

and d_{ij} is defined by

$$d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad 2.7$$

Combining the equations 2.5, 2.6, and 2.7 and substituting into 2.4 yields the vector force equation:

$$\begin{aligned} \rho\left(\frac{\partial \overline{V}}{\partial t}\right) = {} & \rho\overline{F} - \rho(\overline{V}\cdot\nabla)\overline{V} - \nabla p + (\lambda + 2\mu)\nabla(\nabla\cdot\overline{V}) \\ & -\mu\nabla\times(\nabla\times\overline{V}) + (\nabla\cdot\overline{V})\nabla\lambda + 2(\nabla\mu\cdot\nabla)\overline{V} \qquad 2.8 \\ & + \nabla\mu\times(\nabla\times\overline{V}). \end{aligned}$$

To develop the third of the fundamental equations, it is necessary to establish a series of relationships based on the law of conservation of energy. The total energy, E_T, of a given system can be expressed as the sum of the kinetic energy, E_K, and the internal energy, E_I, where

$$E_I \equiv \int_V \rho \varepsilon \, dv \qquad 2.9$$

v - specific volume = $1/\rho$

ε - internal energy density

V - volume of the system.

The material rate of change of the total energy is then the sum of the rates at which work is done upon the system; that is:

$$\frac{D(E_K + E_I)}{Dt} = \int_V \rho F_i v_i \, dv + \oint_S t_{ij} v_j \, dS_i - \oint_S \overline{q}\cdot d\overline{S} \qquad 2.10$$

where

$\overline{q}$ = heat flux vector.

This postulates that the rate of doing mechanical work can be divided into two portions: that arising from the extraneous force and that arising from the forces of material continuity expressed in terms of an equivalent stress vector acting on the surface $\overline{S}$. The third term expresses the rate at which thermal energy enters or leaves the volume.

Applying the divergence theorem and utilizing equation 2.4 reduces 2.10 to

$$\rho\left(\frac{D\varepsilon}{Dt}\right) = t_{\lambda\gamma} d_{\lambda\gamma} - \frac{\partial q_\lambda}{\partial x_\lambda} \qquad 2.11$$

This is the Fourier-Kirchhoff-C. Neumann energy equation. Using the relationships from equations 2.5 and 2.7, 2.11 can be written as

$$\rho\frac{D\varepsilon}{Dt} + \frac{\partial q_\lambda}{\partial x_\lambda} + p\nabla\cdot\overline{V} = u_{\lambda\gamma} d_{\lambda\gamma} \qquad 2.12$$

Considering the internal energy density, ε, to be a function of s and v gives

$$\frac{D\varepsilon}{Dt} = \left(\frac{\partial\varepsilon}{\partial s}\right)_v\left(\frac{Ds}{Dt}\right) + \left(\frac{\partial\varepsilon}{\partial v}\right)_s\left(\frac{Dv}{Dt}\right) \qquad 2.13$$

From the first law of thermodynamics,

$$\left(\frac{\partial\varepsilon}{\partial v}\right)_s = -p \qquad\qquad \left(\frac{\partial\varepsilon}{\partial s}\right)_v = T$$

Thus, equation 2.13 can be written as

$$\rho\frac{D\varepsilon}{Dt} = \rho T\frac{Ds}{Dt} - p\nabla\cdot\overline{V} \qquad 2.14$$

Defining Φ_n, a viscous dissipation function as

$$\Phi_n \equiv u_{\lambda\gamma} d_{\lambda\gamma}$$

and eliminating $\rho\frac{D\varepsilon}{Dt}$ between equations 2.12 and 2.14 gives:

$$\rho T\frac{Ds}{Dt} = \Phi_n - \frac{\partial q_\lambda}{\partial x_\lambda} \qquad 2.15$$

Next, the effects of radiation and conduction must be considered. Since the divergence of $\overline{q}$ gives the energy transferred away from the volume element, it must account for energy loss by either conduction or radiation.

For conduction, the Fourier relation is used:

$$(q_\lambda)_{cond} = K\left(\frac{\partial T}{\partial x_\lambda}\right)$$

where

K - thermal conductivity.

Taking the divergence of each side gives:

$$\frac{\partial(q_\lambda)_{cond}}{\partial x_\lambda} = K\nabla^2 T - (\nabla T)\cdot(\nabla K) \qquad 2.16$$

For the radiation effects, Newton's law of cooling states:

$$\frac{\partial(q_\lambda)_{rad}}{\partial x_\lambda} = \rho C_v \beta(T - T_0) \qquad 2.17$$

where

ϱ - radiation coefficient introduced by Stokes, Phil. Mag. (4) 1, pp 305-317 (1851)

C_v - specific heat at constant volume $\equiv T\left(\frac{\partial s}{\partial T}\right)_v$

$(T - T_0)$ - local temperature excess.

Before proceeding, the term $T\left(\frac{Ds}{Dt}\right)$ appearing in equation 2.15 must be expressed in terms of $\bar{V}$, v, and T. Following the methods of Hunt[15], this is given by

$$\rho T\left(\frac{Ds}{Dt}\right) = \rho C_v(\gamma - 1)\frac{\nabla\cdot\bar{V}}{\beta} + \frac{DT}{Dt} \qquad 2.18$$

$$\gamma \equiv C_p / C_v$$

$$C_p \equiv T(\partial s/\partial T)_p$$

β - coefficient of termal expansion

$$\equiv \rho(\partial v/\partial T)_p$$

Combining equation 2.18 with 2.15, 2.16 and 2.17 gives the third of the required equations:

$$\rho C_v\frac{DT}{Dt} + \rho C_v\left[\frac{\gamma-1}{\beta}\right]\nabla\cdot\bar{V} - K\nabla^2 T \qquad 2.19$$

$$-\nabla T\cdot\nabla K + \rho C_v \varrho (T - T_0) - \Phi_n = 0$$

[15]Hunt, op. cit., pp 1023.

3. Small Signal Acoustic Equations.

The small signal acoustic equations are obtained by arbitrarily "linearizing" the general equations of motion. For this purpose, the dependent variables will be replaced by the sum of their equilibrium or zero-order values and the first-order variational components. Thus,

$$\rho \equiv \rho_0 + \rho_1 \qquad\qquad p \equiv p_0 + p_1$$

$$T \equiv T_0 + \theta_1 \qquad\qquad \overline{V} \equiv 0 + \overline{V}_1$$

On the assumption that ρ_0 , T_0 and p_0 are constants and that $\overline{V}$ is itself a first-order term (i.e. $\overline{V}_0 = 0$) then:

$$\nabla \rho = \nabla \rho_1 \qquad\qquad \nabla p = \nabla p_1$$

$$\nabla T = \nabla \theta_1 \qquad\qquad \nabla \cdot \overline{V} = \nabla \cdot \overline{V}_1$$

Substituting these relations into equations 2.1, 2.8 and 2.19 and retaining only the first-order terms gives:

$$\frac{\partial \rho_1}{\partial t} + \rho_0 (\nabla \cdot \overline{V}_1) = 0 \tag{3.1}$$

$$\rho_0 \left(\frac{\partial \overline{V}_1}{\partial t}\right) + \nabla p_1 - (\mu N) \nabla (\nabla \cdot \overline{V}_1) + \mu \nabla \times (\nabla \times \overline{V}_1) = 0 \tag{3.2}$$

$$\rho_0 C_v \left(\frac{\partial \theta_1}{\partial t}\right) + \left[\rho_0 C_v (\gamma - 1)/\beta\right] (\nabla \cdot \overline{V}_1) - k \nabla^2 \theta_1 - \rho_0 C_v g\, \theta_1 = 0, \tag{3.3}$$

where the extraneous force has been omitted since it primarily affects the equilibrium configuration and

$N = 2 + \lambda/\mu$, the viscosity number.

Considering p as a function of ρ and T gives:

$$\nabla p_1 = \left(\frac{\partial p}{\partial \rho}\right)_T \nabla \rho_1 + \left(\frac{\partial p}{\partial T}\right)_\rho \nabla \theta_1 \tag{3.4}$$

Since:

$$\left(\frac{\partial p}{\partial \rho}\right)_T = \frac{c_0^2}{\gamma}$$

and

$$\left(\frac{\partial p}{\partial T}\right)_\rho = \left(\frac{dv}{d\rho}\right)\left(\frac{\partial v}{\partial T}\right)_\rho = \rho\beta,$$

equation 3.4 becomes

$$\nabla p_1 = \frac{c_0^2}{\gamma} \nabla \rho_1 + \rho_0 \beta \nabla \theta_1$$

and 3.2 can be rewritten as

$$\rho_0\left(\frac{\partial \overline{V}_1}{\partial t}\right) + \left(\frac{c_0^2}{\gamma}\right)\left[1 + \beta\rho_0\left(\frac{\nabla\Theta_1}{\nabla\rho_1}\right)\right]\nabla\rho_1 - \mu N\nabla(\nabla\cdot\overline{V}_1) \qquad 3.5$$
$$+\mu\nabla\times(\nabla\times\overline{V}_1) = 0.$$

These equations (3.1, 3.3, 3.5) take into account the first-order effects of shear and dilatational viscosity and of heat conduction and their solution is discussed in Section 5. They can be reduced to the classical small-signal wave equation by neglecting the effects of heat conduction and viscosity ($K = \varrho = \mu = \lambda = 0$) and assuming adiabatic behaviour of the medium giving

$$\frac{\partial^2 p}{\partial t^2} = c_0^2\nabla^2 p \qquad 3.6$$

4. Solution of the Simple Wave Equation.

Before considering the more complex equations derived in the preceeding section, a solution will first be obtained for the small-signal wave equation. The results obtained will serve as a valuable guide when interpreting the solutions obtained in later sections.

The simple wave equation is given by:

$$\frac{\partial^2 p}{\partial t^2} = c^2 \nabla^2 p \qquad 4.1$$

Where ∇^2 is the Laplacian operator and C is the free space sound velocity. Assuming a time dependence of the form $\varepsilon^{\lambda \sigma t}$ where $\sigma = \omega + \lambda\alpha$, equation 4.1 takes the form of a scalar Helmholtz equation:

$$\left(\nabla^2 + K^2\right) p = 0 \qquad 4.2$$

where

$$K = \frac{\sigma}{c} = \frac{\omega + \lambda\alpha}{c} = K' + \lambda\alpha_1$$

Using the method of separation of variables with the Laplacian operator in cylindrical coordinates gives:

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} + K^2 = 0 \qquad 4.3$$

where R'' denotes $\frac{d^2R}{dr^2}$, Z'' denotes $\frac{d^2Z}{dz^2}$, etc. By the standard argument for the method of separation of variables, equation 4.3 may be rewritten as

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + K^2 = -\frac{Z''}{Z} = K_z^2$$

This then produces two equations:

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + K^2 = K_z^2 \qquad 4.4a$$

$$Z'' + K_z^2 Z = 0 \qquad 4.4b$$

where equation 4.4b is the familiar differential equation of simple harmonic motion with the solution:

$$Z = A'_{K_z}\varepsilon^{\lambda K_z z} + B'_{K_z}\varepsilon^{-\lambda K_z z} \qquad 4.5$$

In general, K_z can be considered complex; that is

$$K_z = \lambda\alpha_2 + K_z'$$

Thus, equation 4.5 may be rewritten as

$$Z = \left(A_{K_z}\cos K_z' z + B_{K_z}\sin K_z' z\right)\varepsilon^{-\alpha_2 z} \qquad 4.5a$$

Equation 4.4a may be separated utilizing the same techniques giving:

$$\Theta'' + m^2 \Theta = 0 \qquad 4.6a$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{1}{r^2}m^2 + K^2 - K_z^2 = 0 \qquad 4.6b$$

where solutions to 4.6 may be written in the form:

$$\Theta = C_m \cos(m\theta + \psi) \qquad 4.7$$

Since Θ must be single valued in θ, m must be an integer.

Equation 4.6b is recognized as a form of Bessel's equation providing a solution (with m as an integer) of the form:

$$R = D_{m,K_r} J_m(K_r r) + E_{m,K_r} N_m(K_r r) \qquad 4.8$$

where $J_m(K_r r)$ is a Bessel function of the first kind and $N_m(K_r r)$ is a Bessel function of the second kind[16], and $K_r^2 = K^2 - K_z^2$. The boundary conditions which must be satisfied by equations 4.5a, 4.7 and 4.8 are:

a. The pressure is finite throughout the interior of the cylinder.

b. The walls are assumed to be pressure release surfaces requiring that $p = 0$ at the boundaries of the cylinder.

Selecting a coordinate system with $z = 0$ at the bottom of the cylinder, and for a cylinder of radius a and length L, the following conditions apply:

a. Coefficient of the cosine term in equation 4.5a must be zero since p is zero at $z = 0$.

b. Periodicity in z is given by $K_z' = \frac{\ell\pi}{L}$ where $\ell = 1,2,3\text{--}$ since p is zero at $z = L$.

c. The coefficient of the second term of equation 4.8 must be zero since Bessel functions of the second kind cannot be present with p finite at the origin.[17]

d. Since the selection of the coordinate system is purely arbitrary the $\theta = 0$ axis can be positioned so that the phase angle in equation 4.7 is zero ($\psi = 0$) with no loss of generality.

[16] McLachlan, Bessel Functions for Engineers - 2nd Ed., Oxford at the Clarendon Press (1955)

[17] McLachlan, op. cit.

Thus, the solution of equation 4.2 takes the form:

$$p = F_{m,K_z',K_r}\, \varepsilon^{-\alpha_2 z} \sin(K_z' z)\cos(m\theta) J_m(K_r r)\varepsilon^{\lambda\sigma t} \quad 4.9$$

The final condition to be satisfied is that the pressure must be zero at $r = a$. Thus

$$J_m(K_r a) = 0$$

Defining a quantity $\mathcal{S}_{mn}$ such that

$$\mathcal{S}_{mn} = K_r a \quad 4.10$$

then

$$J_m(\mathcal{S})_{\mathcal{S}=\mathcal{S}_{mn}} = 0 \quad 4.11$$

where n is an integer denoting the nth zero of a Bessel function of order m. Further, it can be shown[18] that all roots of equation 4.11 are real; that is, $\mathcal{S}_{mn}$ must be real. Thus

$$K_r = \frac{\mathcal{S}_{mn}}{a} \quad 4.12$$

is real. Since

$$K_r^2 = K^2 - K_z^2$$

then

$$K_z^2 = K^2 - \left(\frac{\mathcal{S}_{mn}}{a}\right)^2$$

Substituting the complex forms for K_z^2 and K^2 gives:

$$-\alpha_2^2 + K_z'^2 + 2\lambda\alpha_2 K_z' = -\alpha_1^2 + K'^2 + 2\lambda\alpha_1 K' - \left(\frac{\mathcal{S}_{mn}}{a}\right)^2 \quad 4.13$$

Equating the real and imaginary parts of equation 4.13 gives:

$$-\alpha_2^2 + K_z'^2 = -\alpha_1^2 + K'^2 - \left(\frac{\mathcal{S}_{mn}}{a}\right)^2 \quad 4.14a$$

$$\alpha_2 K_z' = \alpha_1 K' \quad 4.14b$$

Combining equations 4.14a and 4.14b and solving for K' gives:

$$K' = \pm\sqrt{K_z'^2 + \left(\frac{\mathcal{S}_{mn}}{a}\right)^2 - \alpha_1^2\left[\left(\frac{\mathcal{S}_{mn}}{a}\right)^2 \Big/ K_z'^2 + \alpha_1^2\right]} \quad 4.15$$

If α_1, is assumed negligable,

$$K' = \sqrt{K_z'^2 + \left(\frac{\mathcal{S}_{mn}}{a}\right)^2} \quad 4.16$$

[18] A. Gary and G. B. Mathews, A Treatise on Bessel Functions, Macmillan & Co. Ltd. St. Martin's Street, London, pp 79, 1931.

where the negative root is rejected since only positive frequency is being considered. As might be expected, equation 4.16 is the equation for the wave number in a lossless medium. Substituting for K' and K_z' in equation 4.16 gives:

$$\frac{\omega_{\ell mn}}{c} = \sqrt{\left(\frac{\ell\pi}{L}\right)^2 + \left(\frac{\varsigma_{mn}}{a}\right)^2}$$

Therefore, $f_{\ell mn}$, the resonant frequency of the cavity is

$$f_{\ell mn} = \frac{c}{2}\sqrt{\left(\frac{\ell}{L}\right)^2 + \left(\frac{\varsigma_{mn}}{\pi a}\right)^2} \qquad 4.17$$

It is desirable to excite a mode of vibration in which the medium has no tangential components of velocity. For a medium having even a slight viscosity, such motion at the walls can introduce a much greater damping of free vibrations than that due to absorption of energy by the medium itself. Thus, only axially symmetric modes will be considered (i.e. m = 0) for the remainder of the analysis. For the lowest axially symmetric mode, $\varsigma_{mn} = \varsigma_{01} = \pi(0.7655)$ and $\ell = 1$. This corresponds to a pressure antinode at the midpoint of the axis of the resonator.

If attenuation is present, its effect will be to lower the resonant frequency slightly as indicated by equation 4.15.

5. Sound Absorption and Dispersion By the Medium.

Investigation of the general mechanism of absorption by the medium will be based on a solution of the first-order equations obtained in section 3. They were

$$\frac{\partial \rho}{\partial t} + \rho_0 (\nabla \cdot \overline{V}) = 0 \qquad (3.1) \qquad 5.1$$

$$\rho_0 C_v \left(\frac{\partial \theta}{\partial t}\right) + \left[\rho_0 C_v (\gamma - 1)/\beta\right] (\nabla \cdot \overline{V}) - K \nabla^2 \theta - \rho_0 C_v g \theta = 0 \qquad (3.3) \qquad 5.2$$

$$\rho_0 \left(\frac{\partial \overline{V}}{\partial t}\right) + \frac{C_0^2}{\gamma}\left[1 + \beta \rho_0 \left(\frac{\nabla \theta}{\nabla \rho}\right)\right] \nabla \rho - \mu N \nabla (\nabla \cdot \overline{V}) + \mu \nabla \times (\nabla \times \overline{V}) = 0 \qquad (3.5) \qquad 5.3$$

where the subscripts indicating first-order effects have been omitted.

Differentiating equation 5.3 with respect to time and substituting from 5.1 for $\frac{\partial \rho}{\partial t}$ gives

$$\rho_0 \frac{\partial^2 \overline{V}}{\partial t^2} - \frac{\rho_0 C_0^2}{\gamma} \nabla (\nabla \cdot \overline{V}) + \frac{C_0^2}{\gamma} \beta \rho_0 \nabla \frac{\partial \theta}{\partial t} - \mu N \nabla \left(\nabla \cdot \frac{\partial \overline{V}}{\partial t}\right) + \mu \nabla \times \left(\nabla \times \frac{\partial \overline{V}}{\partial t}\right) = 0 \qquad 5.4$$

Considering that, according to Helmholtz's theorem[19] any vector can be uniquely separated into the sum of a gradient of a scalar and the curl of a divergenceless vector, let

$$\overline{V} = \nabla \phi + \nabla \times \overline{A} \qquad 5.5$$

where

ϕ - velocity potential

$\overline{A}$ - vector potential

$\nabla \cdot \overline{A} = 0$

Substituting equation 5.5 into 5.4 yields the following scalar and vector wave equations:

$$\rho_0 \frac{\partial^2 \phi}{\partial t^2} - \frac{\rho_0 C_0^2}{\gamma} \nabla^2 \phi + \frac{C_0^2}{\gamma} \beta \rho_0 \frac{\partial \theta}{\partial t} - \mu N \nabla^2 \left(\frac{\partial \phi}{\partial t}\right) = 0 \qquad 5.6a$$

$$\rho_0 \frac{\partial \overline{A}}{\partial t} + \mu \nabla \times \nabla \times \overline{A} = 0 \qquad 5.6b$$

It is interesting to note that equation 5.6b which governs first-order vorticity is not a true vector wave equation but a vector analogue of the

[19]Morse and Feshback, Methods of Theoretical Physics, Vol. 1, McGraw-Hill Book Co., pp. 53, 1953.

typical homogeneous diffusion equation. Since no source terms are present in 5.6b, this implies that, if first-order vorticity exists, it must be generated at the boundaries. This lack of source terms is due to the technique utilized in generating the first-order equations. Based on the assumption that the equilibrium values of the state variables are constant and that the viscosity coefficients are functions of these state variables, then it is consistant to assume that $\nabla \mu_0 = \nabla \lambda_0 = 0$. Therefore, the source terms involving the gradients of the viscosity coefficients appear as second-order rather than first-order sources. This section deals specifically with absorption by the medium; thus, solution of equation 5.6b and its implications will be reserved for section 6.

Neglecting the effects of radiation and assuming a time dependence of the form $\varepsilon^{i\omega t}$, equations 5.2 and 5.6a become:

$$\nabla^2\phi\left[\frac{\rho_0 C_0^2}{\gamma} + i\omega\mu N\right] + \omega^2\rho_0\phi - i\omega\frac{C_0^2}{\gamma}\beta\rho_0\theta = 0 \qquad 5.7$$

$$\nabla^2\phi\left[\rho_0 C_v(\gamma-1)/\beta\right] - K\nabla^2\theta + i\omega\rho_0 C_v\theta = 0 \qquad 5.8$$

Solving 5.7 for θ and substituting into 5.8, and introducing the shorthand notation:

$$G = \frac{N\mu\omega}{\rho_0 C_0^2}, \text{ a frequency dependent term}$$

and

$$H = \frac{K}{\mu N C_p}, \text{ which is non-frequency sensitive}$$

where

$$GH = \frac{K\omega}{\rho_0 C_0^2 C_p}$$

gives

$$\phi + \frac{C_0^2}{\omega^2}\left[1 + iG(1+\gamma H)\right]\nabla^2\phi - \left[\frac{C_0^4}{\omega^4}G(\gamma GH - iH)\right]\nabla^4\phi = 0 \qquad 5.9$$

Equation 5.9 is biquadratic in the complex propagation constant and it is convenient to rewrite it as

$$(\nabla^2 + K_1^2)(\nabla^2 + K_2^2)\phi = 0 \qquad 5.10$$

The standard quadratic formula can be used to determine the reciprocal squares of the propagation constants as

$$\left.\begin{matrix}\dfrac{1}{K_1^2}\\[2ex] \dfrac{1}{K_2^2}\end{matrix}\right\} = \frac{-C_0^2}{2\omega^2}\left\{1 + iG(1+\gamma H) \pm \left[1 + iG(1-\gamma H)\right]\left[1 + \frac{4i(\gamma-1)GH}{\left[1+iG(1-\gamma H)\right]^2}\right]^{\frac{1}{2}}\right\} \qquad 5.11$$

For the frequency range under consideration with water as the medium, G and H are small numbers and equation 5.11 can be expanded by power series methods to yield:

$$K_1 = \frac{\Lambda\omega}{C_0}\left\{1 + \frac{G^2}{4}\left[\frac{3}{2} + H(\gamma-1)\right] - \frac{\Lambda}{2}G\left[1 + H(\gamma-1)\right]\right\} \qquad 5.12a$$

$$K_2 = \frac{\omega}{C_0}\left\{\frac{1+\Lambda}{(2GH)^{1/2}}\left[1 + \frac{1}{2}(\gamma-1)(1-\gamma H)G\right]\right\} \qquad 5.12b$$

Neglecting second-order terms gives:

$$K_1 = \frac{\Lambda\omega}{C_0}\left\{1 - \frac{\Lambda}{2}\left[\frac{N\nu}{\rho_0} + (\gamma-1)\frac{K}{\rho_0 C_p}\right]\frac{\omega}{C_0^2}\right\} \qquad 5.13a$$

$$K_2 = (1+\Lambda)\left[\frac{\rho_0 C_p \omega}{2K}\right]^{1/2}\left[1 + \frac{1}{2}(\gamma-1)\left(1 - \frac{\gamma K}{\nu N C_p}\right)\left(\frac{N\nu\omega}{\rho_0 C_0^2}\right)\right] \qquad 5.13b$$

Referring again to equation 5.10, we see that the solution will take the form of two waves referred to as types I and II whose wave numbers are respectively K_1 and K_2. Examination of equations 5.13 indicates that the two solutions might also be classified as "compressional" and "thermal" since the type I waves represent the solution to the ordinary wave equation with a slight damping coefficient and the type II waves are generally the result of thermal conduction.

From equation 5.13a the absorption and dispersion measures for the type I waves are:

$$\alpha = \frac{\omega^2}{2\rho_0 C_0^3}\left[\nu N + \frac{(\gamma-1)}{C_p}K\right]$$

$$\left(\frac{C}{C_0}\right)^2 \doteq 1 + \frac{3}{4}\left(\nu\omega N/\rho_0 C_0^2\right)^2$$

$$C \doteq C_0\left[1 + \frac{3}{8}\left(\nu\omega N/\rho_0 C_0\right)^2\right]$$

The attenuation thus determined is the Kirchhoff approximation for low frequencies.

The wave length of the type II wave ($\lambda \doteq 2\pi\left(2K/\rho_0 C_p \omega\right)^{1/2}$) is very much shorter than that of the type I wave. Further, the type II waves are very rapidly attenuated at the frequencies under consideration with a decrement of $\delta \doteq \frac{\lambda}{2\pi}$, and the velocity of propagation through the medium is given by:

$$c^2 \doteq \frac{2\omega K}{\rho_0 C_p}$$

6. Wall Losses.

To study the effects of wall losses on the system, consider the force equation derived in section 2 (2.9). The viscosity coefficients μ and λ are assumed constant, and effects of the extraneous force, F, are assumed negligible. Further, since only small amplitude vibrations are to be considered, all terms involving squares of the velocity components are neglected. Equation 2.8 then becomes

$$\rho\left(\frac{\partial \overline{V}}{\partial t}\right) = -\nabla p + (\lambda + \mu)\nabla(\nabla\cdot\overline{V}) + \mu\nabla^2\overline{V} \qquad 6.1$$

Since it was shown in section 5 that the thermal waves are rapidly attenuated, the effect of heat conduction at the boundary will be neglected. Thus, before proceeding further in the analysis, a direct relationship between pressure and velocity must be established[20]. Flow of mass out of any volume element reduces the pressure in a compressible fluid. In an elastic fluid, if compression is small, the rate of change of pressure is proportional to the divergence of the velocity; that is

$$\frac{\partial p}{\partial t} = -B\,\nabla\cdot\overline{V} \qquad 6.2$$

B - bulk modulus of elasticity

Taking the time derivitive of 6.1 and introducing the expression given in equation 6.2 for $\frac{\partial p}{\partial t}$ gives:

$$\rho\left(\frac{\partial^2\overline{V}}{\partial t^2}\right) = \nabla(B\nabla\cdot\overline{V}) + (\mu+\lambda)\nabla\nabla\cdot\left(\frac{\partial\overline{V}}{\partial t}\right) + \mu\nabla^2\frac{\partial\overline{V}}{\partial t} \qquad 6.3$$

Again defining $\overline{V}$ as

$$\overline{V} = \nabla\phi + \nabla\times\overline{A} \qquad 6.4$$

and substituting equation 6.4 into 6.3 gives the two equations:

$$\rho\frac{\partial\overline{A}}{\partial t} = -\mu\nabla\times\nabla\times\overline{A} \qquad 6.5a$$

$$\rho\frac{\partial^2\phi}{\partial t^2} = B\nabla^2\phi + (2\mu+\lambda)\nabla^2\frac{\partial\phi}{\partial t} \qquad 6.5b$$

Assuming a time dependence of the form $\varepsilon^{\lambda\sigma t}$ where $\sigma = \omega + \lambda\alpha$, equations 6.5a and 6.5b take the form:

[20]Morse and Feshbach, Methods of Theoretical Physics, Vol. 1, McGraw-Hill Book Co., pp. 162, 1953.

$$\nabla \times \nabla \times \bar{A} + h^2 \bar{A} = 0 \qquad 6.6a$$

$$(\nabla^2 + K^2)\,\phi = 0 \qquad 6.6b$$

where

$$h^2 = \frac{\lambda \rho \sigma}{\mathcal{N}}$$

$$K^2 = \frac{\rho \sigma^2}{B + \lambda\sigma(2\mathcal{N} + \lambda)}$$

Since ϕ satisfies a scalar Helmholz equation, the form of solution will be the same as equation 4.9, that is

$$\phi = \sin K_z z \; J_m(K_r r) \begin{matrix}\cos \\ \sin\end{matrix} (m\,\theta)\; \varepsilon^{\lambda \sigma t} \qquad 6.7$$

where

$$K^2 = K_z^2 + K_r^2$$

$$K_z = \frac{l\pi}{L} \qquad l = 1, 2, 3 \cdots$$

$$J_m(K_r a) = 0$$

Equation 6.6a takes the general form of a vector "wave" equation. According to Stratton[21] the three independent vector solutions to 6.6a are given by:

$$\bar{L} = \nabla \Psi \qquad 6.8$$

$$\bar{M} = \nabla \times \bar{a}\,\Psi \qquad 6.9$$

$$\bar{N} = \frac{1}{h} \nabla \times \bar{M} \qquad 6.10$$

where $\bar{a}$ is any constant vector of unit length and Ψ is a scalar function satisfying the equation:

$$\nabla^2 \Psi + h^2 \Psi = 0 \qquad 6.11$$

From 6.8, 6.9 and 6.10,

$$\nabla \cdot \bar{M} = \nabla \cdot \bar{N} = 0$$

$$\nabla \cdot \bar{L} = \nabla^2 \Psi$$

Since, by definition, $\nabla \cdot A = 0$, 6.9 and 6.10 must be the complete solution of 6.6a; that is,

$$\bar{A} = \bar{M} + \bar{N}$$

[21] J. A. Stratton, Electromagnetic Theory, McGraw-Hill Book Co., pp 393, 1941.

Equation 6.11 is again a scalar Helmholz equation and thus Ψ is given by:

$$\Psi = \frac{\cos}{\sin} bz \frac{\cos}{\sin} m\theta \; J_m(gr)$$

Considering only the zero mode (m = 0)

$$\Psi = \frac{\cos}{\sin} bz \; J_0(gr) \qquad 6.12$$

Substituting equation 6.12 into 6.9 and 6.10 and selecting $\bar{a} = \bar{a}_z$ gives:

$$\bar{M} = \nabla \times \bar{a}_z \Psi$$

$$= \bar{a}_\theta g \frac{\cos}{\sin} bz \; J_1(gr) \qquad 6.13$$

$$\bar{N} = \frac{1}{h}(\nabla \times \bar{M})$$

$$= \frac{1}{h}\left\{\bar{a}_r gb \frac{\sin}{-\cos} bz \; J_1(gr) + \bar{a}_z g^2 J_0(gr) \frac{\cos}{\sin} bz\right\} \qquad 6.14$$

Taking the curl of 6.13 and 6.14 gives

$$\nabla \times \bar{A} = \bar{a}\left[bg \frac{\sin}{-\cos} bz \; J_1(gr)\right] \qquad 6.15$$

$$+ h\,\bar{a}_\phi\left[g \frac{\cos}{\sin} bz \; J_1(gr)\right]$$

$$+ \bar{a}_z\left[g^2 J_0(gr) \frac{\cos}{\sin} bz\right]$$

From the symmetry of the problem:

$$V_r(r,z) = V_r(r, L-z) \qquad 6.16a$$

$$V_z(r,z) = -V_z(r, L-z) \qquad 6.16b$$

Thus, the rotational component of the vector velocity has the form

$$\nabla \times \bar{A} = \bar{a}_r\left[C_{b,g}(bg \sin bz \; J_1(gr)\right]$$

$$+ \bar{a}_\phi\left[D_{b,g}(hg \cos J_1(gr)\right] \qquad 6.17$$

$$+ a_z\left[C_{b,g}(g^2 J_0(gr) \cos bz\right]$$

where

$$b^2 + g^2 = h^2$$

$$b^2 = \ell\pi/L \qquad \ell = 1, 2, 3 \cdots$$

From an examination of equations 6.7 and 6.17, it can be seen that since $K_z = b$, then $g \neq K_r$. Thus $J_0(ga) \neq 0$.

The irrotational component of velocity is determined by taking the gradient of 6.7. Thus

$$\nabla\phi = \bar{a}_r\left[E_{K_z,K_r}(-K_r \sin K_z z \; J_1(K_r r)\right] \qquad 6.18$$

$$+ \bar{a}_z\left[E_{K_z,K_r}(K_z \cos K_z z \; J_0(K_r r)\right]$$

A rational accounting of the actual wall losses is well beyond the scope of this discussion. Derivation of even a reasonable approximation for the pressure field existing at the boundary would be difficult and tedious. Even if these equations were known, the mechanism of acoustical energy transfer to, and the absorption by the compliant structure is not sufficiently understood to make application feasible. Thus, it is necessary to assume a simple model, considering that the solutions obtained will act as a guide in interpreting the actual behavior of the resonator.

As a first approximation, the pressure release boundary condition is retained with the additional assumption that the cylinder is bounded by a smooth wall. It is further assumed that, for a viscous medium, there is no slipping between the medium and the wall. This implies the condition that the tangential components of velocity will be zero at the boundary.

Examination of equation 6.18 indicates that the irrotational components already satisfy this condition, i.e.,

$$V_r = 0 \quad \text{at} \quad z = 0,\ z = L$$

$$V_z = 0 \quad \text{at} \quad r = a\,.$$

Considering the rotational components, since $J_1(ga) \neq 0$, $D_{b,g}$ must be zero so that

$$V_\phi = 0 \quad \text{at} \quad r = a$$

Further, $C_{b,g}$ must equal zero for the longitudinal component, V_z, to satisfy the boundary condition. Thus, for this simple model, there is no rotational velocity component, and no first-order vorticity exists. Therefore, the solution obtained in section 4 is unaltered.

A closer approximation to the actual conditions can be obtained by assuming that pressure approaches some small finite value rather than zero at the boundary (i.e., $J_0(K_r a) \neq 0$, $\sin K_z L = 0$). In this case, the coefficient $D_{b,g}$ in equation 6.17 must again be equal to zero. However, from the two conditions:

$$\sum V_z = 0 \quad \text{at} \quad r = a$$

$$\sum V_r = 0 \quad \text{at} \quad z = 0,\ z = L$$

two equations are obtained:

$$E\left[-K_r \sin K_z z\, J_1(K_r r)\right] + C\left[bg \sin bz\, J_1(gr)\right] = 0 \qquad 6.19$$

at $z = 0$ and $z = L$, and

$$E\left[K_z \cos K_z z\, J_0(K_r a)\right] + C\left[g^2 \cos bz\, J_0(ga)\right] = 0. \qquad 6.20$$

Since equation 6.19 must hold for all values of r, it follows that $K_z = b = \frac{\pi l}{L}$. Equation 6.20 can then be rewritten as

$$J_0(K_r a) = -\frac{C}{E}\frac{g^2}{K_z} J_0(ga)$$

$$= C' J_0(ga). \qquad 6.21$$

Equation 6.21 again indicates that for perfect pressure release walls (i.e., $J_0(K_r a) = 0$) there is no first-order vorticity and no viscous loss of energy at the boundary.

As the boundary deviates from the ideal case, the coefficient C' in equation 6.21 increases accordingly and vorticity is generated. Thus, friction arises between the layer of fluid adhering to the walls and the immediately adjoining liquid in motion causing a dissipation of energy at the walls.

This effect can be evaluated by considering that

$$K_r a = \varsigma_{mn} - \varepsilon_{mn}$$

where ε_{mn} is a small complex quantity which is dependent on C'. Referring to equation 4.13, it can be seen that the addition of ε_{mn} will result in a decrease in the resonant frequency of the cavity as well as an increase in the total energy dissipation.

A further implication of this analysis is that, since $P_1 \neq 0$ at $r = a$, a transmitted as well as a reflected wave exists at the boundary and some energy will be lost to the surrounding medium due to radiation.

7. Measurement of Attenuation.

Before proceeding to apply the results obtained in the preceding sections to the actual design of a cavity, some method of measuring the energy absorption by the enclosed medium must be determined. The quality factor or Q of a resonant cavity may be defined as:

$$Q = \frac{2\pi \text{ (energy stored in cavity)}}{\text{energy lost per cycle}} \qquad 7.1$$

The mechanical energy stored in an incremental volume can be considered as the sum of the kinetic and potential energies; that is,

$$E_T = E_K + E_P \qquad 7.2$$

where

$$E_K \equiv \frac{\rho \bar{V}^2}{2} dV$$

$$E_P \equiv p_1^2/2\rho c^2 \, dV$$

Therefore, W, the total energy stored in the cavity, is determined by taking the volume integral of equation 7.2; i.e.,

$$W = \int_V \frac{\rho \bar{V}^2}{2} dV + \int_V \frac{p_1^2}{2\rho c^2} dV \qquad 7.3$$

For the present, wall losses are neglected and all attenuation is assumed to occur in the medium. Selecting a time dependence of the form $\varepsilon^{\wedge\sigma t}$ where σ is again equal to $\wedge\alpha + \omega$ and substituting into equation 7.3 indicates that W decays at a rate determined by $\varepsilon^{-2\alpha t}$ where α is the attenuation constant in nepers per second. Thus, the energy lost during one period is given by:

$$\text{energy lost per cycle} = \frac{2\alpha W}{f} \qquad 7.4$$

Substituting equations 7.3 and 7.4 into 7.1 gives

$$Q = \frac{2\pi f W}{2\alpha W} = \frac{\omega}{2\alpha}$$

or

$$\alpha = \frac{\omega}{2Q} \quad \text{nepers/second} \qquad 7.5$$

To convert α from a time to a space dependence, equation 7.5 is multiplied by $1/c$:

$$\alpha' = \frac{\omega}{2cQ} = \frac{K'}{2Q} \quad \text{nepers/unit length} \qquad 7.6$$

where K' is defined by equation 4.15 as

$$K' = \sqrt{K_z'^2 + \left(\frac{\delta_{mn}}{a}\right)^2 - \alpha^2\left[\left(\frac{\delta_{mn}}{a}\right)^2 / K_z'^2 + \alpha^2\right]}$$

Substituting in equation 7.6 and solving for α gives:

$$\alpha = \frac{1}{2Q}\sqrt{K_z'^2 + \left(\frac{\delta_{mn}}{a}\right)^2}$$

$$= \frac{\pi}{2Q}\sqrt{\left(\frac{\ell}{L}\right)^2 + \left(\frac{\delta_{mn}}{\pi a}\right)^2} \qquad 7.7$$

Thus, with the assumption that all energy dissipation occurs in the medium, the quality factor of the cavity can be utilized as a measure of the attenuation constant of the medium.

In the case of an actual resonator, the effect of wall loss cannot be neglected, and the cavity is characterized by a total Q, Q_T, given by

$$\frac{1}{Q_T} = \frac{1}{Q_m} + \frac{1}{Q_w}$$

where Q_m is due to dissipation by the medium and Q_w is a correction due to wall loss. Q_m, then, is the value which must be utilized in equation 7.7 to measure absorption by the medium.

Since the device under consideration will be operating at relatively low frequencies, determination of the correction factor, Q_w, is greatly simplified.

It is well known that, at frequencies below 10 k.c./sec, attenuation of sound waves in fresh water is considerably less than attenuation of sound in sea water. In fact, by comparison, attenuation by the fresh water can be neglected. Thus, if the cavity is immersed in distilled water and the total quality factor is determined, then

$$\frac{1}{Q_{T_d}} = \frac{1}{Q_{w_d}}$$

to a close approximation. The subscript d is used to indicate measurement in distilled water.

Q may also be defined as

$$Q = \frac{f}{\Delta f}$$

where

f - resonant frequency

$\Delta f = f_2 - f_1$

f_2, f_1 - frequencies of the upper and lower half power points.

Therefore,

$$\frac{1}{Q_m} = \left(\frac{\Delta f}{f}\right)_T - \left(\frac{\Delta f_d}{f_d}\right)_W$$

Considering that the resonant frequencies in fresh and salt water will be only slightly different,

$$\frac{1}{Q_m} = \frac{\Delta f - \Delta f_d}{f}$$

where

Δf - band width between half power points for actual cavity in sea water

Δf_d - band width between half-power points in distilled water

f - resonant frequency in sea water.

Substituting this value into equation 7.7 gives

$$\alpha = \frac{\pi(\Delta f - \Delta f_d)}{2f}\sqrt{\left(\frac{\ell}{L}\right)^2 + \left(\frac{S_{mn}}{\pi a}\right)^2} \qquad 7.8$$

and equation 7.8 can be used to experimentally determine attenuation by the medium.

8. Cavity Design.

In the design and construction of a sound-velocity and attenuation meter based on the principles discussed in the foregoing analysis, two general problems must be considered:

a. Design of a cylindrical cavity presenting essentially a pressure-release boundary over the frequency range under consideration.

b. Development of the "intelligence" system including a transducer to excite the water column to resonance, a receiving transducer to sample the generated pressure field and the associated electronic equipment for interpretation and display of the desired information.

As a first step in the design, the general usage of the data obtained from the instrument must be considered to determine the desired resonant frequency of the cavity. Equation 4.17 can then be utilized to calculate the approximate dimensions of the cylinder. Care must be taken to select a length to diameter ratio that will minimize danger of mode skipping.

Next, the general range of sound velocities to be measured must be specified. This, in turn, determines how broad the Q of the cavity must be to present adequate sensitivity at the extremes of the operating range.

With these parameters in hand, the formulas developed by Toulis[22] can be utilized to determine the nature and spacing of the compliant elements for construction of the cavity walls. This, then, outlines the steps to be carried out in the solution of the first portion of the design problem.

In selecting and mounting the transducers for exciting the water column into resonance and sampling the generated pressure field, several factors must be considered. First of all, since the material generally used for enclosing the transducers is highly absorptive, it is desirable to mount them in a relatively low intensity sound field. Further, it is necessary to keep separation sufficient to minimize direct coupling.

The design of a system for interpreting and presenting the data obtained is primarily dependent upon the actual operating requirements and will not be discussed.

[22] W. J. Toulis, Acoustic Refraction and Scattering With Compliant Elements, to be published.

9. Conclusions.

Examination of the foregoing analysis indicates that the solution to the simple wave equation obtained in Section 4 describes, to a very close approximation, the conditions existing in an actual resonator. From Section 5, it can be seen that the effects of heat conduction on the resonant frequency and attenuation constant are negligible. Section 7 indicates that, with walls closely approximating an ideal pressure-release surface, the effects of first-order vorticity can also be neglected. Thus, for design of a system employing the techniques under discussion, the simplified analysis presented in Section 4 should prove adequate.

One particularly desirable feature of this system is that it measures velocity and attenuation simultaneously. This should be of great assistance in determining the effect of air bubbles and other natural phenomena on sound velocity.

Another desirable feature of this equipment is the frequency range employed. Since some doubt exists as to the validity of adapting sound velocity data measured at megacycle frequencies to much lower-frequency applications, an accurate device operating at these lower frequencies should be a welcome addition.

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APPENDIX I

CAVITY CONSTRUCTION AND TEST

A cylindrical acoustic resonator sound-velocity meter was constructed by Toulis and preliminary tests were conducted by Toulis and the author from 6 January 1957 to 15 March 1957. This appendix outlines the methods of construction and instrumentation utilized and describes the test results.

The pressure-release walls of the cavity under test were constructed by deforming commercially available aluminum tubing (Alcoa 3003-H-14, 1/2 hard, 1" O.D., 0.036" wall thickness) into nearly flat or elliptic cross-section and welding the ends to prevent flooding. These tubes were then secured to a cylindrical frame with 2-1/2 inches spacing between tubes forming a closed cylinder.

The water column was excited into resonance by a cylindrical barium titanate transducer mounted just below the compliant tube top of the cavity near the cylinder wall, and a similar transducer utilized to detect the pressure field was mounted directly opposite.

For making velocity measurements, the output of the receiving transducer was amplified and applied to the exciting transducer. A self-excited oscillating loop was thus formed. The loop voltage was applied, through a frequency multiplier (X10), to a Potter 150 KC. Frequency Counter.

For attenuation measurements, the amplitude of oscillation is an indirect measure of Q. Direct determination was made by driving the cavity with a General Radio Beat Frequency Oscillator (Type 1304-A) and measuring the bandwidth between the half-power points.

During the tests, the output of the receiving transducer was applied to the amplifier through a calibrated potentiometer. The amplifier output was monitored and the gain (potentiometer) setting required to maintain constant output was recorded as an indication of the attenuation. A block diagram of the system is shown in Figure 1.

To evaluate the ability of the instrument to measure actual sound velocity at sea, a number of tests were conducted in the coastal waters of San Diego, California. For comparison purposes and, particularly, to isolate the effects of salinity, an additional set of runs was made at

the NEL testing facility at Sweetwater Lake. At each coastal station, the cavity was lowered by means of a bathythermograph winch, and the following data was recorded as the chamber was stopped at intervals on the ascent:

a. Resonant frequency as indicated by the Potter counter.

b. Temperature as determined by a thermistor bead attached to the side of the cavity.

c. Gain setting.

d. Depth as indicated by a wire metering device.

The results of these tests[23] indicate that the system tested presents promise for development as a survey instrument for making sound-velocity profiles.

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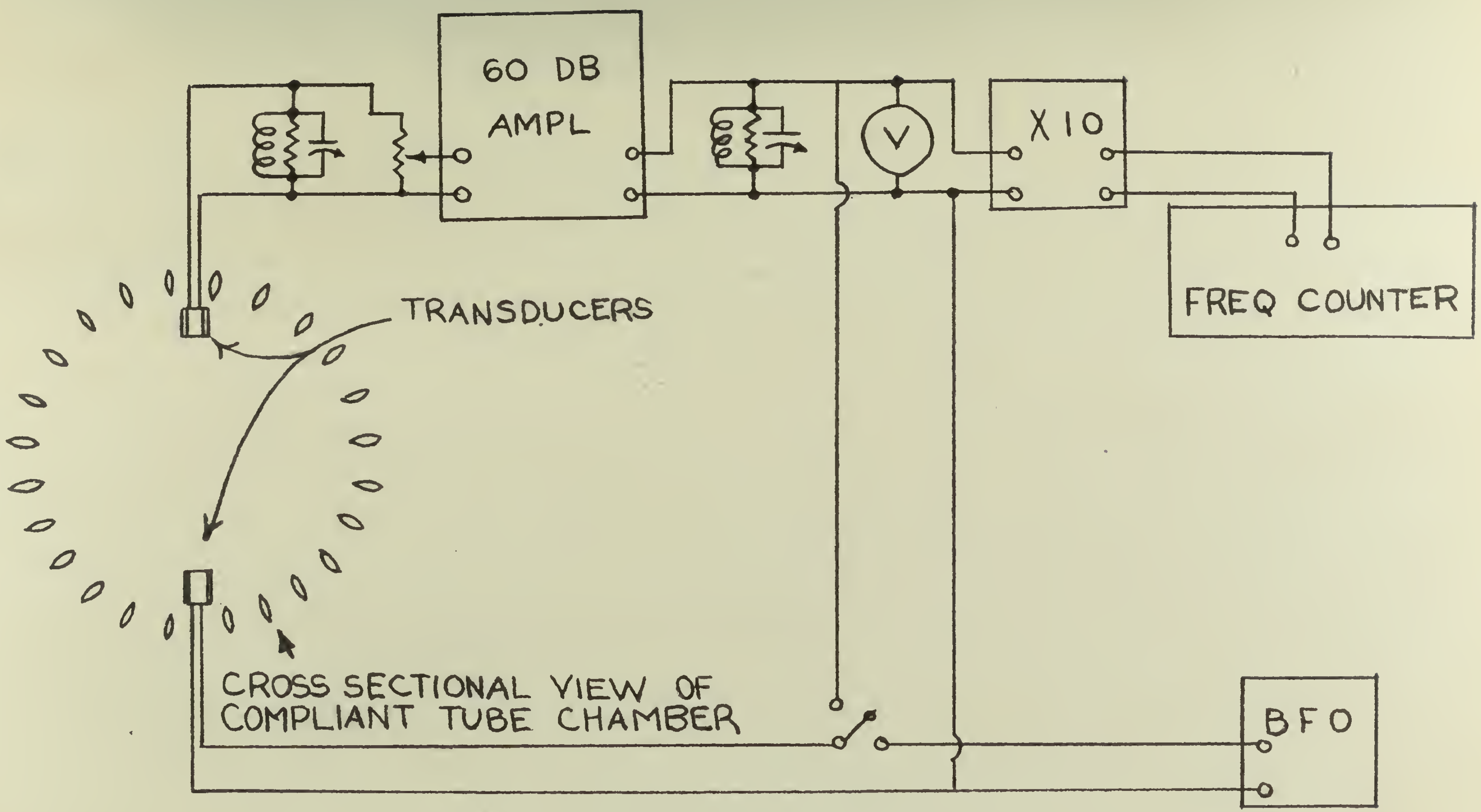

Fig.1 – Block Diagram of Toulis Sound Velocity Meter